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Overlapping Schwarz waveform relaxation method for the solution of the reaction–diffusion equation

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Abstract

We are interested in solving time dependent problems using domain decomposition methods. In the classical approach, one discretizes first the time dimension and then one solves a sequence of steady problems by a domain decomposition method. In this article, we treat directly the time dependent problem and we study a Schwarz waveform relaxation algorithm for the convection diffusion equation. We study the convergence of the overlapping Schwarz waveform relaxation method for solving the reaction–diffusion equation over multi-overlapped subdomains. Also we will show that the method converges linearly and superlinearly over long and short time intervals, and the convergence depends on the size of overlap. Numerical results are presented from solutions of a specific model problems to demonstrate the convergence, linear and superlinear, and the role of the overlap size.

Keywords: Reaction–diffusion equation; Overlapped subdomains; Overlapping Schwarz waveform relaxation method

1. Introduction

Overlapping Schwarz waveform relaxation is the name of the method that utilizes the concepts of the Schwarz iterative process and the waveform relaxation algorithm to solve parabolic equation. The method was introduced independently by Gander and Stuart [5] and Giladi and Keller [7] to solve parabolic equations in parallel over multi-overlapped spatial subdomains.

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The classical approach of applying the domain decomposition methods to solve the parabolic equation is to discretize the equation with respect to the time variable using implicit scheme and to apply the domain decomposition method at each time step. The decomposition of the spatial domain could be of overlapping or nonoverlapping type. The solution of the parabolic problem is obtained by concurring the solution of the subproblems defined over the overlapping or nonoverlapping subdomains.

For nonoverlapping subdomains an interesting noniterative method was proposed by Dawson et al. [3] using explicit prediction for the interface boundary condition followed by implicit solution over the associated subdomains, see also [2]. The main aspect in this approach is that we should use uniform time discretization spacing over the entire domain resulting in the loss of one of the domain decomposition method's main features because the problem is solved differently and independently on each subdomain.

Further research and studies have been carried on the class of noniterative methods to solve nonoverlapping domain decomposition problem. In [14] the authors reported on a class of stabilized explicit–implicit domain decomposition (SEIDD) methods for the numerical solution of parabolic equations. The algorithm presents a stabilization for the Explicit–Implicit domain decomposition (EIDD) method. The EIDD is globally noniterative, for nonoverlapping domain decomposition applications. Also, in [13] a class of corrected explicit–implicit domain decomposition (CEIDD) methods for the parallel approximation of linear heat equations is presented. The method is also a noniterative method and based on the Explicit–Implicit domain decomposition (EIDD).

In the classical waveform relaxation method the iteration is performed using the splitting of the matrix of coefficient associated with the spatial discretization. But, in the overlapping Schwarz waveform relaxation method we use the splitting of the differential operator over the overlapped subdomains [8–10].

Overlapping Schwarz waveform relaxation solves the parabolic problem on overlapped subdomains using Schwarz iteration converging to the solution of the original problem defined on the whole domain. In 1997 Gander [4] and in 1999 Gander and Stuart [5] studied the linear and superlinear convergence of overlapped Schwarz waveform relaxation for solving nonlinear reaction–diffusion and heat equations, respectively. In [4] the author presented the convergence analysis for the linear and superlinear convergence of the reaction–diffusion equation with reaction term being *embedded* within a general uniformly bounded function, defined as source function, bounded by a positive constant. But in this work our analysis and study are presented for the reaction–diffusion equation with reaction coefficient presented by general positive function of the dependent variables. The linear and superlinear convergence of multi-dimensional heat equation was studied by Gander and Zhao [6] and the multi-dimensional parabolic equation was studied by Daoud and Gander [1].

The most recent study in this field is the article by Martin [11]. The author presented new approach to solve two-dimensional convection–diffusion equation by treating, directly, the time dependent problem and studied the convergence of the Schwarz waveform relaxation algorithm. The interface boundary conditions have been defined, through the operators on the interfaces, to minimize the convergence rate.

In this article we shall study the linear and superlinear convergence of overlapping Schwarz waveform relaxation method for solving specific type of reaction–diffusion equation on overlapped subdomains to show how the convergence depends on the size of the overlapped subdomains. In Section 2 we formulate the overlapping Schwarz waveform relaxation algorithm for the reaction–diffusion equation. In Section 3 we prove linear and superlinear convergence over long and short time domains, respectively. We end the article in Section 4 with conclusions and comments from the numerical solution of a specific model problem which confirms the convergence.

2. Overlapping Schwarz waveform relaxation for reaction–diffusion equation

In this section we study the solution of the following reaction–diffusion equation over the domain $\Omega = [0, L] \times [0, T)$

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - r^2(x,t)u, && 0 < x < L,\ 0 < t < T, \\ &u(0,t) = g_1(t), && 0 < t < T, \\ &u(L,t) = g_2(t), && 0 < t < T, \\ &u(x,0) = u_0(x), && 0 < x < L, \end{aligned} \tag{1}$$

where $g_1(t)$, $g_2(t)$, $u_0(x)$ are piecewise continuous functions and $r^2(x,t)$ is bounded from below $r_1^2 < r^2(x,t)$ for $x \in [0, L]$, and $t \in (0, T)$. Then by the existence and uniqueness theorem, Eq. (1) has a solution [12].

To set up the overlapping Schwarz waveform relaxation algorithm for model problem (1), we start by constructing waveform relaxation over two overlapped subdomains $\Omega_1 = [0, L_1] \times [0, T)$ and $\Omega_2 = [L_2, L] \times [0, T)$ where $0 < L_2 < L_1 < L$. The solution of (1) will be given by the solution of the following subproblems:

$$\begin{aligned} &\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - r^2(x,t)v, && 0 < x < L_1,\ 0 < t < T, \\ &v(0,t) = g_1(t), && 0 < t < T, \\ &v(L_1,t) = w(L_1,t), && 0 < t < T, \\ &v(x,0) = u_0(x), && 0 < x < L_1, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} - r^2(x,t)w, && L_2 < x < L,\ 0 < t < T, \\ &w(L_2,t) = v(L_2,t), && 0 < t < T, \\ &w(L,t) = g_2(t), && 0 < t < T, \\ &w(x,0) = w_0(x), && L_2 < x < L, \end{aligned} \tag{3}$$

where $v(x,t) = u(x,t)|_{\Omega_1}$ and $v(x,t) = u(x,t)|_{\Omega_2}$.

To find the solution of (2) and (3) we use waveform relaxation of the Schwarz type iterative method as follows:

$$\begin{aligned} &\frac{\partial v^{k+1}}{\partial t} = \frac{\partial^2 v^{k+1}}{\partial x^2} - r^2(x,t)v^{k+1}, && 0 < x < L_1,\ 0 < t < T, \\ &v^{k+1}(0,t) = g_1(t), && 0 < t < T, \\ &v^{k+1}(L_1,t) = w^k(L_1,t), && 0 < t < T, \\ &v^{k+1}(x,0) = u_0(x), && 0 < x < L_1, \end{aligned} \tag{4}$$

and

$$
\begin{aligned}
&\frac{\partial w^{k+1}}{\partial t} = \frac{\partial^2 w^{k+1}}{\partial x^2} - r^2(x,t)w^{k+1}, && L_2 < x < L,\ 0 < t < T,\\
&w^{k+1}(L_2,t) = v^k(L_2,t), && 0 < t < T,\\
&w^{k+1}(L,t) = g_2(t), && 0 < t < T,\\
&w^{k+1}(x,0) = u_0(x), && L_2 < x < L.
\end{aligned} \tag{5}
$$

To investigate the convergence of the iterative solutions of subproblems (4) and (5) we define the errors $d^{k+1}(x,t) = u(x,t)|_{\Omega_1} - v^{k+1}(x,t)$ and $e^{k+1}(x,t) = u(x,t)|_{\Omega_2} - w^{k+1}(x,t)$ on subdomains Ω_1 and Ω_2, respectively. The error equations corresponding to $d^{k+1}(x,t)$ and $e^{k+1}(x,t)$ are given by

$$
\begin{aligned}
&\frac{\partial d^{k+1}}{\partial t} = \frac{\partial^2 d^{k+1}}{\partial x^2} - r^2(x,t)d^{k+1}, && 0 < x < L_1,\ 0 < t < T,\\
&d^{k+1}(0,t) = 0, && 0 < t < T,\\
&d^{k+1}(L_1,t) = e^k(L_1,t), && 0 < t < T,\\
&d^{k+1}(x,0) = 0, && 0 < x < L_1,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
&\frac{\partial e^{k+1}}{\partial t} = \frac{\partial^2 e^{k+1}}{\partial x^2} - r^2(x,t)e^{k+1}, && L_2 < x < L,\ 0 < t < T,\\
&e^{k+1}(L_2,t) = d^k(L_2,t), && 0 < t < T,\\
&e^{k+1}(L,t) = 0, && 0 < t < T,\\
&e^{k+1}(x,0) = 0, && L_2 < x < L.
\end{aligned} \tag{7}
$$

We will present the convergence of the error equations (6) and (7) for two different time intervals: short and long time intervals.

3. Linear and superlinear convergence

In this section we will present the proof of the linear and superlinear convergence of the overlapping Schwarz waveform relaxation algorithm for the solution of the reaction–diffusion equation. For linear convergence we will consider the error equations (6) and (7) defined over long time intervals, i.e. $T = \infty$.

The key theorem for convergence is the positivity lemma stated by Pao [12] for the bounded time domain. In [4] Gander presented the proof of the positivity lemma for the unbounded time interval.

Lemma 3.1 *(Positivity Lemma [4]). Assume that the function* $w \in C([0,L] \times [0,\infty) \cap C^{2,1}((0,L) \times (0,\infty))$ *satisfies the differential inequalities*

$$
\begin{aligned}
&\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + r^2(x,t)w \geqslant 0, && 0 < x < L,\ t > 0,\\
&w(0,t) \geqslant 0, && t > 0,\\
&w(L,t) \geqslant 0, && t > 0,\\
&w(x,0) \geqslant 0, && 0 \leqslant x \leqslant L,
\end{aligned} \tag{8}
$$

where $r^2(x,t)$ is bounded from below, i.e. $r_1^2 < r^2(x,t)$ for some constant r_1^2, for all $x \in (0,L)$ and $t \in (0,\infty)$. Then

$$w(x,t) \geqslant 0,$$

for all $x \in [0,L]$ and $t \in (0,\infty)$.

Proof. For the proof see [4]. □

To show linear convergence of (6) and (7) we shall consider the infinity norm defined by

$$\|f(.,.)\|_\infty = \sup_{x\in(0,L),t>0} |f(t)|,$$

for all functions $f \in L^\infty = L^\infty(\mathbf{R}^+, \mathbf{R})$.

Theorem 3.2. *Suppose that the function $r^2(x,t)$ is bounded from below, i.e. $0 < r_1^2 < r^2(x,t)$ for all $x \in [0,L]$ and $t \in (0,\infty)$. Then the errors d^{k+2} and e^{k+2} of the iterations* (6) *and* (7) *on the interfaces at $x = L_1$ and $x = L_2$ decay linearly. In particular,*

$$\|d^{k+2}(L_2,.)\|_\infty \leqslant \Gamma(L_1,L_2)\|d^k(L_2,.)\|_\infty \tag{9}$$

and

$$\|e^{k+2}(L_1,.)\|_\infty \leqslant \Gamma(L_1,L_2)\|e^k(L_1,.)\|_\infty, \tag{10}$$

where

$$\Gamma(L_1,L_2) = \frac{\sinh(L_1 r_1)}{\sinh(L_2 r_1)} \frac{\sinh((L-L_2)r_1)}{\sinh((L-L_1)r_1)} < 1.$$

The proof of Theorem 3.2 requires the following lemma.

Lemma 3.3. *Consider the following differential equation*:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - r^2(x,t)u, && 0 < x < L,\ t > 0,\\
&u(0,t) = g_1(t), && t > 0,\\
&u(L,t) = g_2(t), && t > 0,\\
&u(x,0) = 0, && 0 < x < L.
\end{aligned} \tag{11}$$

If $r^2(x,t)$ is bounded from below by r_1^2, i.e. $0 < r_1^2 < r^2(x,t)$, then the solution $u(x,t)$, of (11)*, satisfies the following inequality*:

$$\|u(x,.)\|_\infty \leqslant \frac{\sinh((L-x)r_1)}{\sinh(Lr_1)}\|g_1(.)\|_\infty + \frac{\sinh(xr_1)}{\sinh(Lr_1)}\|g_2(.)\|_\infty. \tag{12}$$

Proof. Consider the differential equation

$$\begin{aligned}
&\frac{\partial \hat{u}}{\partial t} = \frac{\partial^2 \hat{u}}{\partial x^2} - r_1^2 \hat{u}, && 0 < x < L,\ t > 0,\\
&\hat{u}(0,t) = \|g_1(.)\|_\infty, && t > 0,\\
&\hat{u}(L,t) = \|g_2(.)\|_\infty, && t > 0,\\
&\hat{u}(x,0) = \frac{\sinh((L-x)r_1)}{\sinh(Lr_1)}\|g_1(.)\|_\infty + \frac{\sinh(xr_1)}{\sinh(Lr_1)}\|g_2(.)\|_\infty, && 0 < x < L.
\end{aligned} \tag{13}$$

The solution $\hat{u}$ of (13) is given by

$$\hat{u}(x) = \frac{\sinh((L-x)r_1)}{\sinh(Lr_1)} \|g_1(.)\|_\infty + \frac{\sinh(xr_1)}{\sinh(Lr_1)} \|g_2(.)\|_\infty.$$

Let $\bar{u}(x) = \hat{u} - u$, therefore $\bar{u}$ satisfies the differential inequality

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial x^2} - r_1^2 \hat{u} + r^2(x,t)u, && 0 < x < L,\ t > 0,\\
&u(0,t) \geqslant 0, && t > 0,\\
&u(L,t) \geqslant 0, && t > 0,\\
&u(x,0) \geqslant 0, && 0 < x < L.
\end{aligned} \tag{14}$$

To apply the Positivity Lemma 3.1 and concluding (12), we rewrite the right-hand side of (14) as follows:

$$\begin{aligned}
\frac{\partial \bar{u}}{\partial t} &= \frac{\partial^2 \bar{u}}{\partial x^2} - r_1^2 \hat{u} + r^2(x,t)u \\
&= \frac{\partial^2 \bar{u}}{\partial x^2} - r_1^2 \hat{u} + r^2(x,t)u + r^2(x,t)\hat{u} - r^2(x,t)\hat{u}.
\end{aligned} \tag{15}$$

From (15), we deduce that

$$\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2} + r^2(x,t)\bar{u} = -\big(r_1^2 - r^2(x,t)\big)\hat{u}. \tag{16}$$

Hence, since the solution $\hat{u}$ is nonnegative and $r^2(x,t)$ is bounded from below by r_1^2, then the first term on the right of (16) is nonnegative and therefore the partial differential equation in (14) can be replaced by the following differential inequality:

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2} + r^2(x,t)\bar{u} \geqslant 0, && 0 < x < L,\ t > 0,\\
&\bar{u}(0,t) \geqslant 0, && t > 0,\\
&\bar{u}(L,t) \geqslant 0, && t > 0,\\
&\bar{u}(x,0) \geqslant 0, && 0 < x < L.
\end{aligned}$$

Then by the Positivity Lemma 3.1 we conclude that $\bar{u} = \hat{u} - u \geqslant 0$ for all $(x,t) \in \Omega$.

Therefore

$$\|u(x,.)\|_\infty \leqslant \frac{\sinh((L-x)r_1)}{\sinh(Lr_1)} \|g_1(.)\|_\infty + \frac{\sinh(xr_1)}{\sinh(Lr_1)} \|g_2(.)\|_\infty. \qquad \square$$

To prove Theorem 3.2 we will apply Lemma 3.3 to the two recursive errors d^{k+2} and e^{k+2} associated with the solution of subproblems (6) and (7) over Ω_1 and Ω_2, respectively.

Proof of Theorem 3.2. Consider the recursive error differential equations (6) and (7). Then by Lemma 3.3 the errors d^{k+2} and e^{k+2} are given by

$$\|d^{k+2}(x,.)\|_\infty \leqslant \frac{\sinh(xr_1)}{\sinh(L_1 r_1)} \|e^{k+1}(L_1,.)\|_\infty \tag{17}$$

and

$$\left\|e^{k+1}(x,.)\right\|_\infty \leqslant \frac{\sinh((L-x)r_1)}{\sinh((L-L_2)r_1)}\left\|d^k(L_2,.)\right\|_\infty. \tag{18}$$

By successive substituting of (17) and (18) evaluated at L_2 and L_1, respectively,

$$\left\|d^{k+2}(L_2,.)\right\|_\infty \leqslant \frac{\sinh(L_2r_1)}{\sinh(L_1r_1)}\frac{\sinh((L-L_1)r_1)}{\sinh((L-L_2)r_1)}\left\|d^k(L_2,.)\right\|_\infty \tag{19}$$

and

$$\left\|e^{k+2}(L_1,.)\right\|_\infty \leqslant \frac{\sinh((L-L1)r_1)}{\sinh((L-L_2)r_1)}\frac{\sinh(L_2r_1)}{\sinh(L_1r_1)}\left\|e^k(L_1,.)\right\|_\infty. \tag{20}$$

Therefore

$$\Gamma(L_1,L_2)=\frac{\sinh((L-L_1)r_1)}{\sinh((L-L_2)r_1)}\frac{\sinh(L_2r_1)}{\sinh(L_1r_1)}<1. \qquad \square \tag{21}$$

From Theorem 3.2 we conclude that the convergence of (6) and (7) depends on the size of the overlap $L_1 - L_2$, and the error decays faster for larger overlaps.

Figure 1 shows how the convergence factor $\Gamma(L_1,L_2)$ ($\Gamma(L_1,L_2)<1$) varies for different overlap size over the spatial interval $0<x<1$, and from it we can deduce the effect of overlap size on the decay of the errors d^{k+2} and e^{k+2}.

In the remaining part of this section we will study the superlinear convergence of the iterations (6) and (7) defined over bounded time interval $[0,T)$, $T<\infty$. We consider the infinity norm defined by

$$\|f(.)\|_T = \sup_{t\in(0,T)}\left|f(t)\right|,$$

for all functions $f \in L^\infty([0,T);\mathbf{R})$.

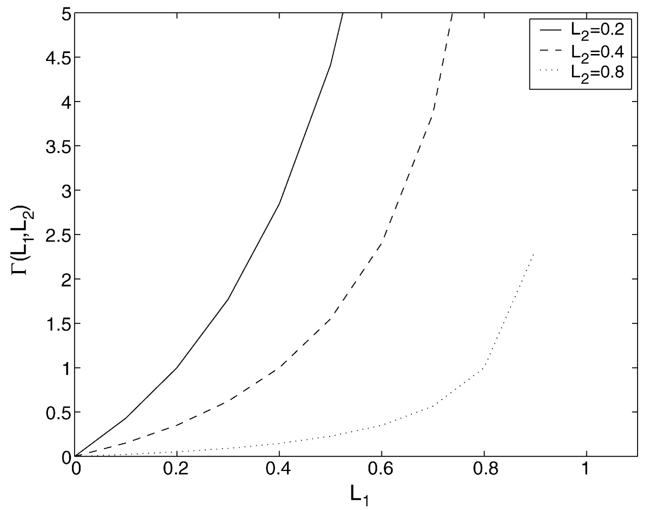

Fig. 1. The theoretical linear convergence factor $\Gamma(L_1,L_2)$ for the solution of the reaction–diffusion equation, for $L_2=0.2, 0.4, 0.8$, and L_1 varying through [0, 1].

The error bound for short time intervals along the interface lines over each of the subdomains of the overlapping Schwarz waveform relaxation algorithm is given by the following theorem.

Theorem 3.4. *Suppose that the reaction coefficient function $r^2(x,t)$ is bounded from below by r_1^2, i.e. $0 < r_1^2 < r^2(x,t)$. Then the errors d^{k+2} and e^{k+2} in the iterations* (6) *and* (7) *decay superlinearly on the interfaces $x = L_1$ and $x = L_2$. In particular, d^{k+2} and e^{k+2} are given by*

$$\|d^{2k}(L_2,.)\|_T \leqslant \max\bigl(e^{-r_1^2},1\bigr)\operatorname{erfc}\left(\frac{k(L_1-L_2)}{\sqrt{T}}\right)\|d^0(L_2,.)\|_T, \tag{22}$$

$$\|e^{2k}(L_1,.)\|_T \leqslant \max\bigl(e^{-r_1^2},1\bigr)\operatorname{erfc}\left(\frac{k(L_1-L_2)}{\sqrt{T}}\right)\|e^0(L_1,.)\|_T. \tag{23}$$

Proof. Consider the following differential equation defined over bounded time domain $[0,T)$:

$$\begin{aligned}
&\frac{\partial \tilde{d}^{k+2}}{\partial t} = \frac{\partial^2 \tilde{d}^{k+2}}{\partial x^2} - r_1^2\tilde{d}^{k+2}, && 0 < x < L_1,\ 0 < t < T,\\
&\tilde{d}^{k+2}(0,t) = 0, && t \in (0,T),\\
&\tilde{d}^{k+2}(L_1,t) = \max\bigl|e^{k+1}(L_1,t)\bigr|, && t \in (0,T),\\
&\tilde{d}^{k+2}(x,0) = 0, && 0 < x < L_1.
\end{aligned} \tag{24}$$

The solution of (24) is given by

$$\tilde{d}^{k+2}(x,t) = \int_0^t K_x(L_1-x,t-\tau)e^{-r_1^2(t-\tau)}\bigl|e^{k+1}(L_1,\tau)\bigr|\,d\tau, \tag{25}$$

where $K_x(x,t)$ is given by

$$K_x(x,t) = \frac{x}{2\sqrt{\pi}\,t^{3/2}}e^{-\frac{x^2}{4t}}.$$

Consider $\bar{d}^{k+2} = \tilde{d}^{k+2} - d^{k+2}$, then $\bar{d}^{k+2}$ satisfies the following differential inequality:

$$\begin{aligned}
&\frac{\partial \bar{d}^{k+2}}{\partial t} - \frac{\partial^2 \bar{d}^{k+2}}{\partial x^2} + r_1^2\bar{d}^{k+2} \geqslant 0, && 0 < x < L_1,\ 0 < t < T,\\
&\bar{d}^{k+2}(0,t) = 0, && t \in (0,T),\\
&\bar{d}^{k+2}(L_1,t) \geqslant 0, && t \in (0,T),\\
&\bar{d}^{k+2}(x,0) = 0, && 0 < x < L_1.
\end{aligned} \tag{26}$$

Therefore the differential inequality (26) satisfies the necessary conditions of the Positivity Lemma giving $\bar{d}^{k+1} \geqslant 0$ ($d^{k+1} < \tilde{d}^{k+1}$), for all $x \in [0,L]$ and $t \in [0,T)$, so

$$\bigl|d^{k+2}(x,t)\bigr| \leqslant \tilde{d}^{k+2} = \int_0^t K_x(L_1-x,t-\tau)e^{-r_1^2(t-\tau)}\bigl|e^{k+1}(L_1,\tau)\bigr|\,d\tau. \tag{27}$$

From similar arguments we conclude that

$$\bigl|e^{k+1}(x,t)\bigr| \leqslant \int_0^t K_x(x-L_2,t-\tau)e^{-r_1^2(t-\tau)}\bigl|d^k(L_2,\tau)\bigr|\,d\tau. \tag{28}$$

Evaluating (28) at $x = L_1$ and substituting back into (27) gives

$$\begin{aligned}&\left|d^{k+2}(x,t)\right|\\ &\quad\leqslant \int_0^t K_x(L_1-x,t-\tau)e^{-r_1^2(t-\tau)}\int_0^\tau K_x(L_1-L_2,\tau-s)e^{-r_1^2(\tau-s)}\left|d^k(L_2,s)\right|ds\,d\tau. \qquad (29)\end{aligned}$$

Then by induction and evaluating (29) at $x = L_2$ we conclude that

$$\begin{aligned}&\left|d^{2k}(L_2,t)\right|\\ &\quad\leqslant \int_0^t K_x(L_1-L_2,t-s_1)e^{-r_1^2(t-s_1)}\dots\\ &\qquad\times\int_0^{s_{2k-1}} K_x(L_1-L_2,s_{2k-1}-s_{2k})e^{-r_1^2(s_{2k-1}-s_{2k})}\left|d^0(L_2,s_{2k})\right|ds_{2k}\dots ds_1. \qquad (30)\end{aligned}$$

The exponential terms in (30) are combined as follows

$$e^{-r_1^2(t-s_1)}e^{-r_1^2(s_1-s_2)}\dots e^{-r_1^2(s_{2k-1}-s_{2k})} = e^{-r_1^2(t-s_{2k})},$$

and by simplifying (30) we conclude that

$$\begin{aligned}\left|d^{2k}(L_2,t)\right| &\leqslant \left\|d^0(L_2,.)\right\|\max\bigl(e^{-r_1^2t},1\bigr)\int_0^t K_x(L_1-L_2,t-s_1)\dots\\ &\qquad\times\int_0^{s_{2k-1}} K_x(L_1-L_2,s_{2k-1}-s_{2k})\,ds_{2k}\dots ds_1. \qquad (31)\end{aligned}$$

To proceed in our analysis we will consider the Laplace transform of (31). Firstly the Laplace transform of the kernel $K_x(x,t)$ is given by

$$\int_0^\infty e^{st}K_x\bigl((L_1-L_2),t\bigr)\,dt = e^{-\frac{(L_1-L_2)}{2}\sqrt{s}},$$

and the Laplace transform of the $2k$-fold convolution is given by

$$e^{-k(L_1-L_2)\sqrt{s}}. \qquad (32)$$

By transforming back of (32) then (31) is given by

$$\left|d^{2k}(L_2,t)\right| \leqslant \left\|d^0(L_2,.)\right\|\max\bigl(e^{-r_1^2t},1\bigr)\int_0^t K_x\bigl(2k(L_1-L_2),t-\tau\bigr)\,d\tau. \qquad (33)$$

Therefore the error $d^{2k}(L_2,t)$ (33) is bounded by

$$\left|d^{2k}(L_2,t)\right| \leqslant \left\|d^0(L_2,.)\right\|\max\bigl(e^{-r_1^2t},1\bigr)\operatorname{erfc}\left(\frac{k(L_1-L_2)}{\sqrt{t}}\right). \qquad (34)$$

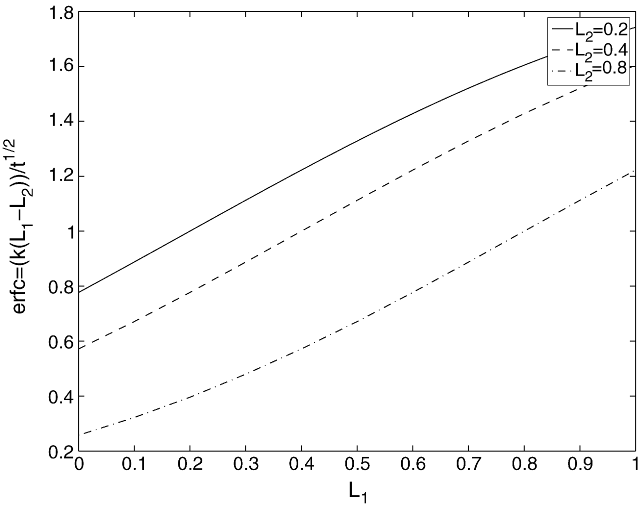

Fig. 2. The superlinear convergence factor $\operatorname{erfc}(\frac{k(L_1-L_2)}{\sqrt{t}})$ for $L_2 = 0.2$, 0.4, 0.8 and L_1 varying through $[0, 1]$ with fixed iteration number $k = 10$ for the solution of the reaction–diffusion equation.

Similarly we may show that

$$\left|e^{2k}(L_1, t)\right| \leqslant \left\|e^0(L_1, .)\right\| \max\left(e^{-r_1^2 t}, 1\right) \operatorname{erfc}\left(\frac{k(L_1 - L_2)}{\sqrt{t}}\right). \qquad \square \tag{35}$$

In Fig. 2 we plot the superlinear theoretical error decay factor $\operatorname{erfc}(\frac{k(L_1-L_2)}{\sqrt{t}})$ for the solution of the reaction–diffusion equation using different subdomains overlaps. Figure 2 shows how convergence varies for different overlap sizes over the spatial domain $0 < x < 1$, and the effect of overlap size on the decay of the errors d^{k+2} and e^{k+2}.

4. Numerical experiments and discussions

In this section we present numerical experiments from the solution of the following reaction–diffusion equation.

Model Problem.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 0.25\pi^2 u, && 0 < x < 1,\ 0 < t < T,\\
&u(0, t) = e^{-0.5\pi^2 t}, && 0 < t < T,\\
&u(1, t) = 0, && 0 < t < T,\\
&u(x, 0) = \cos\left(0.5\pi^2 x\right), && 0 < x < 1.
\end{aligned} \tag{36}$$

Numerical experiments were performed to estimate the convergence rate of the overlapping Schwarz waveform relaxation algorithm and to compare it with the theoretical bounds derived in previous sections for the reaction–diffusion equation.

To solve the model problem we discretized the Laplacian operator using centered finite difference and the backward Euler's in time using different spatial and time spacings. We split the domain $\Omega = [0, 1] \times [0, T)$ into 2, 4 and 5 overlapping subdomains of overlap size 0.2 and 0.4,

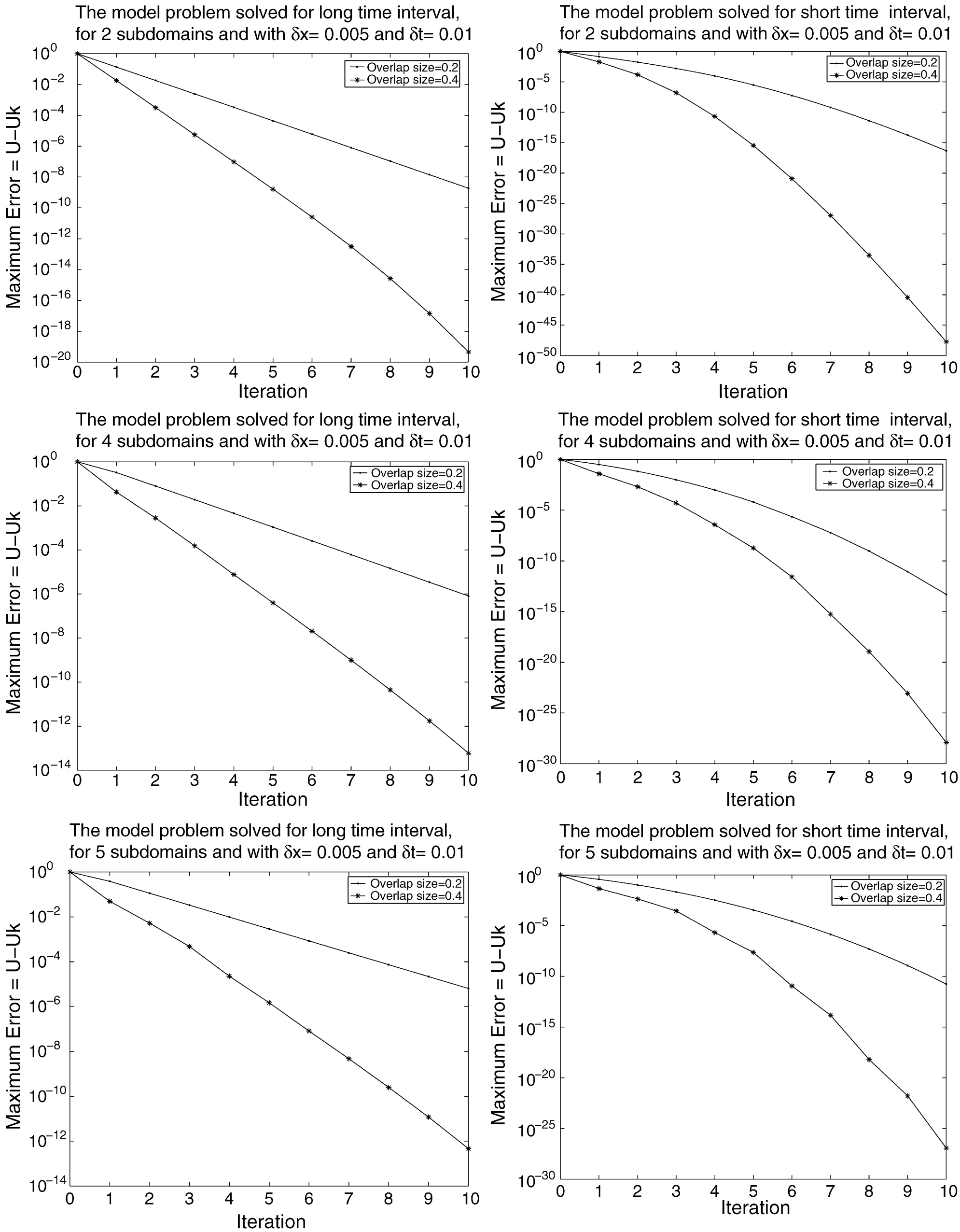

Fig. 3. The error decay of the linear (left) and superlinear (right) convergence for the solution of the reaction–diffusion equation for 2, 4, and 5 subdomains with 0.2 and 0.4 of overlap size.

respectively. We measured the error from the problem with respect to each type of convergence in the infinity norm at the interface points.

We simulated the error equations (6) and (7) for the convergence of the reaction–diffusion equation, and employed homogeneous initial and boundary conditions to converge to the zero solution, respectively.

To observe the linear and superlinear convergence, the model was integrated over long time interval $T = 10$ and short time interval $T = 1$, respectively.

The numerical experiments for the solution of the reaction–diffusion equation, model problem, are presented in Fig. 3. The linear convergence is illustrated by the set of figures on the left while the superlinear convergence is illustrated by the set of figures on the right. The results confirm the conclusions in Theorems 3.2 and 3.4 that the rate of convergence depend on the size of subdomain overlap. Furthermore it is clear that the decay of the error from the simulation over short time interval (superlinear convergence) is faster and gives more accuracy than the simulation over long time interval (linear convergence).

Finally, we found that the error of the linear and superlinear convergence decays faster when the size of the overlap increases.

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